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COMMENT

On the general solution for a 'diagonal' vacuum Bianchi type III model with a cosmological constant

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Abstract. The particular Bianchi type III solution given in a recent letter by Moussiaux *et al* is shown to be contained in a general solution for locally-rotationally-symmetric hypersurface-homogeneous models given by Cahen and Defrise.

One of us (MM) has noted that the particular solution for a 'diagonal' vacuum Bianchi-III model with a cosmological constant, given in a recent letter (Moussiaux *et al* 1981, referred to as MTD), is not new. In fact, the solutions (13a) and (13b) in MTD (respectively for $\Lambda>0$ and $\Lambda<0$), which are locally rotationally symmetric, are particular cases of a solution given in tables 11.1 and 11.2 of Kramer *et al* (1980) (referred to as KSMH): they are characterised geometrically by the action of an isometry group G_4 on a three-dimensional space-like manifold.

More explicitly, solutions (13a) and (13b) in MTD are of the form of (11.3) (with $k=-1$ and $\varepsilon=-1$) of KSMH, i.e.

$$ds^2=-dt^2+A^2\,dx^2+B^2\,(dy^2+\sinh^2 y\,dz^2) \tag{1}$$

with A, B functions of t.

As pointed out by Åman and Karlhede (1980), the coordinates y and z can be transformed into x^1 and x^2, as used in MTD, changing the two-dimensional metric $dy^2+\sinh^2 y\,dz^2$ into $(dx^1)^2+\exp(-4a_0x^1)\,(dx^2)^2$ of equation (8) of MTD.

Moreover, the metric (1) can be put in new coordinates w, u, ζ and $\bar\zeta$ as (cf KSMH, equation (11.11))

$$ds^2=\frac{dw^2}{f(w)}-f(w)\,du^2+\left(\frac{2d\zeta\,d\bar\zeta}{(1-\frac{1}{2}\zeta\bar\zeta)^2}\right)Y^2(w) \tag{2}$$

with $f<0$.

Solutions (13a) and (13b) of MTD can be put in a similar form as

$$ds^2=Y^2(\tau)[-d\tau^2+2\,d\zeta\,d\bar\zeta/(1-\tfrac{1}{2}\zeta\bar\zeta)^2]+b^2(\tau)\,(dx^3)^2 \tag{3}$$

with, in the case of $\Lambda>0$,

$$Y^2=\frac{12a_0^2}{\Lambda\,\{\sinh[2a_0(\tau-\tau_0)]\}^2},\qquad b^2=\{\tanh[2a_0(\tau-\tau_0)]\}^{-2}. \tag{4}$$

The general solution of (2) is given in KSMH as (cf equation (11.42); cf Cahen and Defrise (1968))

$$f(w)=w^{-2}(-w^2-2mw-\tfrac{1}{3}\Lambda w^4), \qquad Y^2(w)=w^2, \qquad \text{and } m=\text{constant}. \tag{5}$$

Identifying now metrics (2) and (3), it is easy to check that metric (2) is of the form (3), with $m=0$ and $u=x^3$ and

$$\begin{aligned} &Y^2=w^2=\frac{12a_0^2}{\Lambda\{\sinh[2a_0(\tau-\tau_0)]\}^2}, \\ &\frac{2\sqrt{3}a_0}{\sqrt{\Lambda}\sinh[2a_0(\tau-\tau_0)]}\mathrm{d}\tau=-\frac{\mathrm{d}w}{(1+\frac{1}{3}\Lambda w^2)^{1/2}}, \\ &b^2=-f. \end{aligned} \tag{6}$$

The solution (13*b*) of MTD can be identified similarly.

On the other hand, choosing A_1 as the new time variable in the general 'diagonal' Bianchi-III metric (equation (8) of MTD), the new form of this metric is

$$\mathrm{d}s^2=-F^2(\mathrm{d}A_1)^2+(A_1)^2(\mathrm{d}x^1)^2+(A_2)^2\exp(-4a_0x^1)(\mathrm{d}x^2)^2+A_3^2(\mathrm{d}x^3)^2 \tag{7}$$

where F, A_2 and A_3 are functions of A_1.

Denoting the first derivative with respect to A_1 by a prime, the corresponding independent field equations can be written as

$$\begin{aligned} &\frac{A_2'}{A_2}-\frac{1}{A_1}=0, \qquad \frac{1}{A_1^2F^2}-\frac{2F'}{A_1F^3}-\frac{4a_0^2}{A_1^2}-\Lambda=0, \\ &\frac{1}{F^2}\left(2\frac{A_3'}{A_1A_3}+\frac{1}{(A_1)^2}\right)-\frac{4a_0^2}{A_1^2}-\Lambda=0. \end{aligned} \tag{8}$$

The general solution of this system of differential equations is easily found as

$$A_2=A_1, \qquad F=\frac{1}{A_3}=\frac{1}{4a_0^2+\frac{1}{3}\Lambda A_1^2+C/A_1} \tag{9}$$

where C is an arbitrary constant, a solution in fact equivalent to the general solution of Cahen and Defrise (1968) (cf (5)).

Since the first of equations (8) implies $A_1=A_2$, it forbids the existence of non-locally-rotationally-symmetric 'diagonal' vacuum Bianchi-III models.

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